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Hurwitz number of triple Ramified covers

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Abstract

Using symplectic cut-and-gluing formulae of the relative Gromov–Witten invariants, we get a recursive formula for the Hurwitz number of triple ramified coverings of a Riemann surface by a Riemann surface.

MSC: 14H10; 57M10; 58F05

Keywords: Hurwitz number; Ramified cover; Symplectic cut; Relative Gromov–Witten invariant

0. Introduction

Let (M, ω) be a $2n$-dimension symplectic manifold with symplectic structure ω, a Hamilton circle action and its moment map $\mu : M \to \mathbb{R}$. The circle action on M can be extended to $M \times \mathbb{C}$ in two ways, $e^{i\theta}(m, w) = (e^{i\theta} \cdot m, e^{-i\theta} \cdot w)$ and $e^{i\theta}(m, w) = (e^{i\theta} \cdot m, e^{i\theta} \cdot w)$. Their two moment maps are Φ_1 and Φ_2, respectively, which are given by the following:

$$\Phi_1,\ \Phi_2 : M \times \mathbb{C} \to \mathbb{R},$$

$\Phi_1(m, w) = \mu(m) - |w|^2$, and $\Phi_2(m, w) = \mu(m) + |w|^2$. Then the symplectic reductions via Φ_1 and Φ_2 yield two symplectic manifolds $\overline{M}_+$ and $\overline{M}_-$, respectively. We call this process symplectic cutting, in Section 1.

In Section 2, we recall the relative Gromov–Witten (GW) invariant [1,8,14]. If the Riemann surface C^g with genus g has c connected components, we can define the relative GW-invariant by the product of c relative Gromov–Witten invariants of connected components.

In Section 3, we recall the Hurwitz Number and introduce the theorem of [17] for 2-simple ramified covering.

Finally, in Section 4, we investigate the 3-simple ramified coverings of Riemann surfaces. By using symplectic cutting and a relative GW-invariant, we induce the following theorem on the Hurwitz number of triple ramified covers.

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Theorem. *All Hurwitz numbers $\mu^{g,k,3}_{h,m}(\alpha)$ of 3-simple ramified coverings can be determined by the following recursive formula:*

$$\begin{aligned}
\mu^{g,k,3}_{h,m}(\alpha) = & \sum_{\theta_1\in J_1(\alpha)} \mu^{g,k,3}_{h,m-2}(\theta_1)\cdot I_1(\theta_1) + \sum_{\theta_2\in C(\alpha)} \mu^{g-1,k,3}_{h,m}(\theta_2)\cdot I_2(\theta_2) \\
& + \sum_{\theta_2\in C(\alpha)} \sum_{\substack{g_1+g_2=g\\ g_1,g_2\ge 0}} \sum_{\substack{k_1+k_2=k\\ k_1,k_2\ge 1}} \sum_{\substack{m_1+m_2=m\\ m_1,m_2\ge 1}} \sum_{\pi\in P_{\theta_2}} \begin{pmatrix} \frac{1}{2}(k+m)-kh+g-2 \\ \frac{1}{2}(k_1+m_1)-k_1h+g_1-1 \end{pmatrix} \\
& \times \mu^{g_1,k_1,3}_{h,m_1}(\theta_{2,\pi_1})\cdot \mu^{g_2,k_2,3}_{h,m_2}(\theta_{2,\pi_2})\cdot I_3(\pi) \\
& + \sum_{\theta_3\in D(\alpha)} \mu^{g-2,k,3}_{h,m+2}(\theta_3)\cdot I_4(\theta_3) \\
& + \sum_{\theta_3\in D(\alpha)} \sum_{\substack{g_1+g_2=g-1\\ g_1,g_2\ge 0}} \sum_{\substack{k_1+k_2=k\\ k_1,k_2\ge 1}} \sum_{\substack{m_1+m_2=m+2\\ m_1,m_2\ge 1}} \sum_{\pi\in P_{\theta_3}} \begin{pmatrix} \frac{1}{2}(k+m)-kh+g-2 \\ \frac{1}{2}(k_1+m_1)-k_1h+g_1-1 \end{pmatrix} \\
& \times \mu^{g_1,k_1,3}_{h,m_1}(\theta_{3,\pi_1})\cdot \mu^{g_2,k_2,3}_{h,m_2}(\theta_{3,\pi_2})\cdot I_5(\pi) \\
& + \sum_{\theta_3\in D(\alpha)} \sum_{\substack{g_1+g_2+g_3=g\\ g_1,g_2,g_3\ge 0}} \sum_{\substack{k_1+k_2+k_3=k\\ k_1,k_2,k_3\ge 1}} \sum_{\substack{m_1+m_2+m_3=m+2\\ m_1,m_2,m_3\ge 1}} \sum_{\pi\in \bar{P}_{\theta_3}} \begin{pmatrix} \frac{1}{2}(k+m)-kh+g-2 \\ \frac{1}{2}(k_1+m_1)-k_1h+g_1-1 \end{pmatrix} \\
& \times \begin{pmatrix} N \\ \frac{1}{2}(k_2+m_2)-k_2h+g_2-1 \end{pmatrix} \mu^{g_1,k_1,3}_{h,m_1}(\theta_{3,\pi_1})\cdot \mu^{g_2,k_2,3}_{h,m_2}(\theta_{3,\pi_2})\cdot \mu^{g_3,k_3,3}_{h,m_3}(\theta_{3,\pi_3})\cdot I_6(\pi) \\
& + \mu^{1,k,3}_{0,3}(k;3,1,\ldots,1;k)\cdot \mu^{g_1,k,3}_{h,m}(\alpha),
\end{aligned}$$

where $N = [\frac{1}{2}(k+m)-kh+g-2] - [\frac{1}{2}(k_1+m_1)-k_1h+g_1-1]$.

1. Symplectic cut

Let (M,ω) be a symplectic manifold with a Hamiltonian circle S^1-action and its moment map $\mu : M \to \mathbb{R}$.

[1.1] If the circle S^1 acts freely on a level set $\mu^{-1}(\epsilon)$, then ϵ is a regular value of the moment map $\Phi_1 : M\times\mathbb{C} \longrightarrow \mathbb{R}$, given by $\Phi_1(m,w) = \mu(m) - |w|^2$, arising from the action of S^1 on the product $(M\times\mathbb{C}, \omega\oplus -\mathrm{i}\mathrm{d}w\wedge\mathrm{d}\overline{w})$, the action being $\mathrm{e}^{\mathrm{i}\theta}(m,w) = (\mathrm{e}^{\mathrm{i}\theta}\cdot m, \mathrm{e}^{-\mathrm{i}\theta}\cdot w)$.

The level set $\{\Phi_1 = \epsilon\}$ is a disjoint union of two S^1-invariant manifolds:

$$\{\Phi_1=\epsilon\} = \{(m,w)\in M\times\mathbb{C} \mid \mu(m)>\epsilon, w=\mathrm{e}^{\mathrm{i}\theta}\sqrt{\mu^{-1}(m)-\epsilon}\} \amalg \{(m,0)\in M\times\mathbb{C} \mid \mu(m)=\epsilon\}$$

which is S^1-equivariantly diffeomorphic to the disjoint union of two S^1-invariant manifolds $\{M_{\mu>\epsilon}\times S^1\}\amalg\mu^{-1}(\epsilon)$.

Thus we have an orbit space which is a symplectic manifold:

$$\begin{array}{lll}
\overline{M}_+ & := & \{\Phi_1=\epsilon\}/S^1 \\
 & \simeq & M_{\mu>\epsilon} \qquad\qquad \amalg \qquad\qquad \mu^{-1}(\epsilon)/S^1 \\
 & & :\ \text{open dense in } \overline{M}_+, \qquad :\ \text{the reduced space,} \\
 & & :\ \text{symplectic submanif.,} \qquad :\ \text{codimen. 2 symplectic submanif.}
\end{array}$$

Topologically, $\overline{M}_+$ is the quotient of the manifold with boundary $M_{\mu\ge\epsilon}$ by the relation $\sim$, where $m\sim m'$ iff $\mu(m)=\mu(m')=\epsilon$ and $m=\mathrm{e}^{\mathrm{i}\theta}\cdot m'$ for some $\mathrm{e}^{\mathrm{i}\theta}\in S^1$.

[1.2] Consider the diagonal action of the circle S^1 on the product $(M\times\mathbb{C}, \omega\oplus i\mathrm{d}w\wedge\mathrm{d}\overline{w})$, i.e., $\mathrm{e}^{\mathrm{i}\theta}(m,w) = (\mathrm{e}^{\mathrm{i}\theta}\cdot m, \mathrm{e}^{\mathrm{i}\theta}\cdot w)$. The corresponding moment map $\Phi_2 : M\times\mathbb{C}\longrightarrow\mathbb{R}$ is $\Phi_2(m,w) = \mu(m)+|w|^2$. Then as above, we have a symplectic manifold:

$$\begin{array}{lcll} \overline{M}_- & := & \{\Phi_2 = \epsilon\}/S^1 & \\ & \overset{S^1\text{-equiv}}{\simeq} & [(M_{\mu<\epsilon} \times S^1) \amalg \mu^{-1}(\epsilon)]/S^1 & \\ & \simeq & M_{\mu<\epsilon} \qquad \amalg & \mu^{-1}(\epsilon)/S^1 \\ & & \text{: open dense in } \overline{M}_+, & \text{: the reduced space,} \\ & & \text{: symplectic submanif.,} & \text{: codimen.2 symplectic manif.} \\ & & & \text{with opposite normal bundle} \\ & & & \text{to the normal bundle in} \overline{M}_+. \end{array}$$

So, the symplectic gluing of $\overline{M}_+$ and $\overline{M}_-$ along the reduced space $\mu^{-1}(\epsilon)/S^1$ recovers the original manifold (M, ω). We call the operation that produces $\overline{M}_+$ and $\overline{M}_-$ symplectic cutting [16–18].

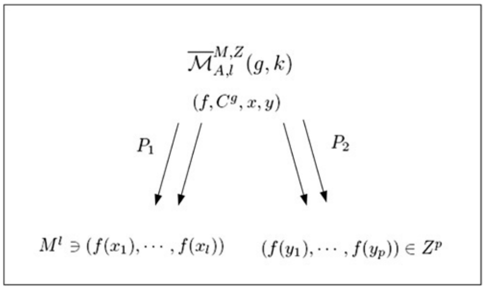
2. Relative Gromov–Witten invariant

Let (M, ω) be a closed symplectic manifold with $A \in H_2(M, \mathbb{Z})$ and $Z \subset M$ be a symplectic submanifold [4–6,8] of M with codimension 2. Let C^g be a (singular) Riemann surface with arithmetic genus g. Let $\overline{\mathcal{M}}^{M,Z}_{A,l}(g, k)$ be the moduli space [15,19] of stable relative pseudo-holomorphic maps $f : C^g \to M$ with marked points $x_1, \ldots, x_l, y_1, \ldots, y_p$ in C^g such that $[f(C^g)] = A$, f tangent to Z at $y_1, \ldots, y_p$ with order $k = (k_1, \ldots, k_p)$.

The virtual dimension of $\overline{\mathcal{M}}^{M,Z}_{A,l}(g, k)$ is

$$\dim \overline{\mathcal{M}}^{M,Z}_{A,l}(g, k) = 2c_1(M) \cdot A + (2n - 6)(1 - g) + 2(l + p) - 2 \cdot \sum_{i=1}^{p} k_i.$$

There are natural evaluation maps:



The relative Gromov–Witten invariant is defined by

$$\psi^{M,Z}_{A,g,l}(\delta, \beta; k) = \int_{\overline{\mathcal{M}}^{M,Z}_{A,l}(g,k)} P_1^*\left(\prod_{i=1}^{l} \delta_i\right) \wedge P_2^*\left(\prod_{i=1}^{p} \beta_i\right),$$

where $\delta_i \in H^*(M)$, $i = i, \ldots, l$, $\beta_i \in H^*(Z)$, $i = 1, \ldots, p$.

The relative GW-invariant $\psi^{M,Z}_{A,g,l}(\delta, \beta; k)$ is defined to be zero, unless

$$\sum_{i=1}^{l} \deg \delta_i + \sum_{i=1}^{p} \deg \beta_i = \dim \overline{\mathcal{M}}^{M,Z}_{A,l}(g, k).$$

If C^g has c connected components $C^{g_1}, \ldots, C^{g_c}$, then the genus $g = \sum_{i=1}^{c} g_i - c + 1$. Let $\mathcal{P}_x$ be the set of all ordered partitions of $\{x_1, \ldots, x_l\}$ into c parts. Each $\pi = (\pi_1, \ldots, \pi_c) \in \mathcal{P}_x$ records which marked points $x = (x_1, \ldots, x_l)$ go on each component $C^{g_1}, \ldots, C^{g_c}$. Similarly, $\sigma = (\sigma_1, \ldots, \sigma_c) \in \mathcal{P}_y$.

Corresponding to the partition of $y = (y_1, \ldots, y_p)$, we can define the partition of k, i.e., $\sigma = (\sigma_1, \ldots, \sigma_c) \in \mathcal{P}_k$. π and σ induce partitions of δ and β, respectively. Denote the parameters over the components C^{g_i} by x_{π_i}, y_{σ_i}, δ_{π_i}, and β_{σ_i}.

Suppose that $f_i : C^{g_i} \to M$ is a relative stable pseudo-holomorphic map such that $[f_i(C^{g_i})] = A_i$ and $A = \sum_{i=1}^{c} A_i$. For $A = \sum_{i=1}^{c} A_i$ and partitions π and σ, the relative GW-invariant is defined by

$$\psi_{A,g,l}^{M,Z,c}(\delta,\beta;k)(\pi,\sigma) = \prod_{i=1}^{c} \psi_{A_i,g_i,l_i}^{M,Z}(\delta_{\pi_i},\beta_{\sigma_i};k_{\sigma_i}).$$

Consider a symplectic cut

$$\begin{array}{ccccc} C^g & \xrightarrow{f} & M & \xrightarrow{\mu} & \mathbb{R} \\ \| & & \downarrow & & \\ C^{g_+} \cup C^{g_-} & \xrightarrow{f^+,f^-} & \overline{M}_+ \cup_B \overline{M}_-, & & \text{where } B = \mu^{-1}(0)/S^1. \end{array}$$

If f^+, f^- have the same orders as the tangent to symplectic submanifold B, by gluing and perturbating, we have a unique pseudo-holomorphic map $f : C^g \to M$.

Consider the moduli space $\mathcal{M}^+ = \mathcal{M}^{+\overline{M}_+,Z^+,B,c^+}_{A^+,l^+}(g^+,k^+,\alpha^+)$ of the tuples $(C^{g^+}, x^+, y^+, e^+, k^+, \alpha^+, f^+)$, where

(1) C^{g^+} has c^+ connected components,
(2) $[f^+(C^{g^+})] = A^+$,
(3) f^+ is tangent to Z^+ at $y^+ = (y_1,\ldots,y_q)$ with order $k^+ = (k_1,\ldots,k_q)$, and
(4) f^+ is tangent to B at $e^+ = (e_1^+,\ldots,e_\nu^+)$ with order $\alpha^+ = (\alpha_1^+,\ldots,\alpha_\nu^+)$.

Similarly, the moduli space $\mathcal{M}^- = \mathcal{M}^{-\overline{M}_-,Z^-,B,C^-}_{A^-,l^-}(g^-,k^-,\alpha^-)$ of the tuples $(C^{g^-}, x^-, y^-, e^-, k^-, \alpha^-, f^-)$.

Let $\mathcal{C}_{g,l,k}^{J,A}$ be the set of indices:

(i) $(C^{g^\pm}, f^\pm), \{A_i^\pm, g_i^\pm, l_i^\pm, k_i^\pm, (\alpha_1^\pm,\ldots,\alpha_\nu^\pm)\}, i = 1,\ldots,c^\pm, \sum_{i=1}^{\nu} \alpha_i^\pm = A \cdot B$,
(ii) $\rho : \{e_1^+,\ldots,e_\nu^+\} \to \{e_1^-,\ldots,e_\nu^-\}$: a 1-1 correspondence in $C^g = C^{g^+} \cup C^{g^-}$, $e_i^- = \rho(e_i^+)$, $g = g^+ + g^- + \nu - 1$, $f^+(e_i^+) = f^-(\rho(e_i^+))$, and $((C^{g^+}, f^+), (C^{g^-}, f^-), \rho)$ represents A.

Now, we introduce the gluing formula of the Gromov–Witten invariant.

Theorem (*[16]*).

(1) *The set* $\mathcal{C}_{g,l,K}^{J,A}$ *is finite,*
(2) $\psi_{A,g,l}^{M,Z}(\delta,\beta;K) = \sum_{C \in \mathcal{C}_{g,l,K}^{J,A}} \psi_C(\delta,\beta;K)$

$$\text{where } \psi_C(\delta,\beta;K) = \|\alpha\| \sum \delta^{IJ} \psi_{A^+,g^+,l^+}^{\overline{M}_+,Z^+,B,c^+}(\delta^+ \mid \beta^+;\rho_I;K^+,\alpha) \times (\pi_C^+,\sigma_C^+) \cdot \psi_{A^-,g^-,l^-}^{\overline{M}_-,Z^-,B,c^-}(\delta^- \mid \beta^-;\rho_J;K^-,\alpha)(\pi_C^-,\sigma_C^-),$$

$\|\alpha\| = \alpha_1 \ldots \alpha_\nu$ *and* $\{\rho_1,\ldots,\rho_s\}$ *is an orthonormal basis of* $H^*(B)$, $\rho_I = \{\rho_{I_1},\ldots,\rho_{I_\nu}\}$, $\rho_J = \{\rho_{J_1},\ldots,\rho_{J_\nu}\} \in \{\rho_1,\ldots,\rho_s\}$.

3. Hurwitz number

[3.1] A ramified covering of a connected Riemann surface Σ^h with genus h of degree k by a connected Riemann surface Σ^g with genus g ($g \geq h \geq 0$) is a non-constant holomorphic map $f : \Sigma^g \to \Sigma^h$ such that $|f^{-1}(q)| = k$ for all but a finite number of points $q \in \Sigma^h$, which are called branch points.

Two ramified coverings $f_1, f_2 : \Sigma^g \to \Sigma^h$ are equivalent if there is a homeomorphism $\pi : \Sigma^g \to \Sigma^g$ such that $f_1 = f_2 \circ \pi$.

A ramified covering f is called n-simple if $|f^{-1}(q)| = k - n + 1$ for each branch point q but one, that is denoted by ∞, and there is a point $x \in f^{-1}(q)$ such that the order of f at x is n.

If $\alpha_1,\ldots,\alpha_m$ are the orders of the preimage of ∞, then the ordered m-tuple $(\alpha_1,\ldots,\alpha_m) = \alpha$ is a partition of k and is called the ramification type [2,3,7] of f at ∞. $l(\alpha) = m$ is called the length of α.

The number $\mu^{g,k,n}_{h,m}(\alpha)$ of equivalent n-simple coverings of Σ^h by Σ^g with ramification type of α is called a Hurwitz number. Hurwitz [13] computed $\mu^{0,k,2}_{0,m}(\alpha)$. Dénes, Arnold and Goulden gave some formulae [9–12].

[3.2] Li, Zhao, and Zheng in [17] produced a formula for "The Number of Ramified covering of a Riemann Surface by a Riemann Surface" as follows.

Theorem *([17]). All $\mu^{g,k,2}_{h,m}(\alpha)$ can be determined by a recursive formula:*

$$\mu^{g,k,2}_{h,m}(\alpha) = \sum_{\theta\in J(\alpha)} \mu^{g,k,2}_{h,m-1}(\theta)\cdot I_1(\theta) + \sum_{\omega\in C(\alpha)} \mu^{g-1,k,2}_{h,m+1}(\omega)\cdot I_2(\omega)$$
$$+ \sum_{\omega\in C(\alpha)} \sum_{\substack{g_1+g_2=g\\ g_1,g_2\geq 0}} \sum_{\substack{k_1+k_2=k\\ k_1,k_2\geq 1}} \sum_{\substack{m_1+m_2=m+1\\ m_1,m_2\geq 1}} \sum_{\pi\in P_\omega} \binom{m+k-2kh-3+2g}{k_1+m_1-2k_1h-2+2g_1}$$
$$\times \mu^{g_1,k_1,2}_{h,m_1}(\omega_{\pi_1})\cdot \mu^{g_2,k_2,2}_{h,m_2}(\omega_{\pi_2})\cdot I_3(\pi).$$

Here,

$$\alpha = (\alpha_1,\ldots,\alpha_m),$$
$$\theta = (\alpha_1,\ldots,\hat{\alpha_i},\ldots,\hat{\alpha_j},\ldots,\alpha_m,\alpha_i+\alpha_j),$$
$$\omega = (\alpha_1,\ldots,\hat{\alpha_i},\ldots,\alpha_m,\rho,\alpha_i-\rho),\ \text{and}$$
$$I_1(\theta) = \begin{cases} \frac{1}{2}(\alpha_i+\alpha_j)\cdot \sharp\{\lambda\in\theta|\lambda=\alpha_i+\alpha_j\} & \text{if } \alpha_i=\alpha_j \\ (\alpha_i+\alpha_j)\cdot \sharp\{\lambda\in\theta|\lambda=\alpha_i+\alpha_j\} & \text{if } \alpha_i\neq\alpha_j, \end{cases}$$
$$I_2(\omega) = \begin{cases} \frac{1}{2}\rho(\alpha_i-\rho)\cdot \sharp\{\lambda\in\omega|\lambda=\rho\}\cdot \sharp(\{\mu\in\omega|\mu=\alpha_i-\rho\}-1) & \text{if } \rho=\alpha_i-\rho \\ \rho(\alpha_i-\rho)\cdot \sharp\{\lambda\in\omega|\lambda=\rho\}\cdot \sharp\{\mu\in\omega|\mu=\alpha_i-\rho\} & \text{if } \rho\neq\alpha_i-\rho, \end{cases}$$
$$I_3(\pi) = \begin{cases} \frac{1}{2}\rho(\alpha_i-\rho)\cdot \sharp\{\lambda\in\omega_{\pi_1}|\lambda=\rho\}\cdot \sharp\{\mu\in\omega_{\pi_2}|\mu=\alpha_i-\rho\} & \text{if } \rho=\alpha_i-\rho \\ \rho(\alpha_i-\rho)\cdot \sharp\{\lambda\in\omega_{\pi_1}|\lambda=\rho\}\cdot \sharp\{\mu\in\omega_{\pi_2}|\mu=\alpha_i-\rho\} & \text{if } \rho\neq\alpha_i-\rho, \end{cases}$$

where $\pi=(\pi_1,\pi_2)$, $\omega=(\omega_{\pi_1},\omega_{\pi_2})$ is a partition of ω, $\rho\in\omega_{\pi_1}$, $\alpha_i-\rho\in\omega_{\pi_2}$.

They use the relative GW-invariants and a gluing formula to induce the recursive formula.

4. Triple Ramified coverings of Riemann surfaces

Let Σ^h be a Riemann surface with genus h. Consider a 3-simple ramified covering holomorphic map $f:\Sigma^g\to\Sigma^h$ of degree k.

[4.1] Apply the relative GW-invariant to Σ^h.

Let $H_2(\Sigma^h,\mathbb{Z})=\mathbb{Z}=H^2(\Sigma^h,\mathbb{Z})=\langle H\rangle$, $c_1(\Sigma^h)=(2-2h)H$, $A=kH$, and $Z=\{Z_0,\ldots,Z_p\}\subset\Sigma^h$. An element $f\in\mathcal{M}^{\Sigma^h,Z}_{A,l}(g,k)$ means that $f:\Sigma^g\to\Sigma^h$ is a pseudo-holomorphic map such that $[f(\Sigma^g)]=kH$ with marked points $x,y\in\Sigma^g$, f is tangent to Z at y with order $k=(k_0,\ldots,k_p)$, $\deg(f)=k$, and $k_i=\sum_{j=1}^{l_i}k_i^j$ is a partition of k for each $i=0,\ldots,p$.

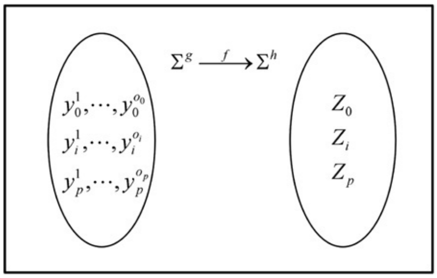
The relative GW-invariant $\psi^{\Sigma^h,Z}_{A,g,0}(0,\beta;k)=0$, unless $\sum_{i=0}^{p}\deg\beta_i = 2c_1(\Sigma^h)\cdot A+4(g-1)+2\sum_{i=0}^{p}l_i - 2\sum_{i=0}^{p}\sum_{j=1}^{l_i}k_i^j = \dim\overline{\mathcal{M}}^{\Sigma^h,Z}_{A,l}(g,k)$.

If $k=(\alpha_1,\ldots,\alpha_m;3,1,\ldots,1;\ldots;3,1,\ldots,1)$, then

$$\dim\overline{\mathcal{M}}^{\Sigma^h,Z}_{A,l}(g,k) = 2c_1(\Sigma^h)\cdot A+4(g-1)+2[m+p(k-2)]-2k(p+1)$$
$$= 2[k+m-2hk+2g-2-2p],$$

where p is the number of triple ramification points.

Since $Z=(Z_0,\ldots,Z_p)$ is $(p+1)$-points in Σ^h, $\deg\beta_i=0$ for $i=0,\ldots,p$.



Proposition. *Suppose that Σ^g is connected, $l = 0$, and $k = (\alpha_1, \ldots, \alpha_m; 3, 1, \ldots, 1; \ldots; 3, 1, \ldots, 1)$. Then*

(1) *the Hurwitz number $\mu_{h,m}^{g,k,3}(\alpha)$ equals to the GW-invariant $\psi_{A,g,0}^{\Sigma^h,Z}(0, \beta; k)$.*

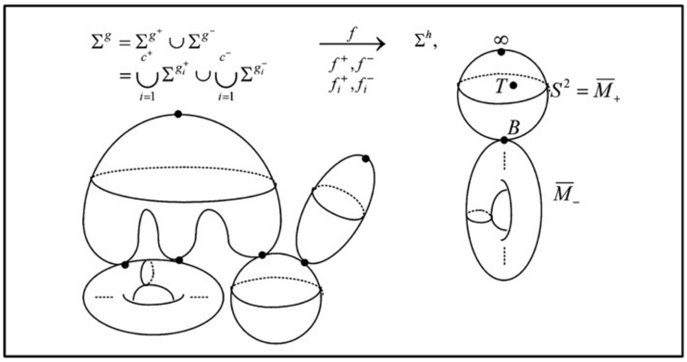
(2) $2p = k + m - 2hk + 2g - 2$, *otherwise $\psi_{A,g,0}^{\Sigma^h,Z}(0, \beta; k) = 0$, where p is the number of triple ramification points.*

[4.2] Perform a symplectic cut over Σ^h at $\infty = Z_0$ in a small neighbourhood such that there is only one other triple branch point T in this neighbourhood. Then we have $\overline{M}_+ = S^2$, $\overline{M}_- \simeq \Sigma^h$ and $A^+ = kH'$, $A^- = kH$, $H_2(S^2, \mathbb{Z}) = \langle H' \rangle$. Dimension of the moduli space:

$$\begin{cases} (k - m) + (k - (k - 2)) + (k - \nu) = 2k - 2 + 2g^+, \\ g = g^+ + g^- + \nu - 1, \end{cases}$$

where ν is the number of the points where we glue Σ^{g^+} and Σ^{g^-}, and $f^+ : \Sigma^{g^+} \to \overline{M}_+$ branches at only 3 points: ∞, T and the symplectic reduction point B.

Suppose that $\Sigma^{g^+} = \coprod_{i=1}^{c^+} \Sigma^{g_i^+}$ has c^+ connected components and the map f has a following type: $f_i^+ : \Sigma^{g_i^+} \to \overline{M}_+$ has degree $k_i^+ \leq k^+ \leq k$;



(A) If $\Sigma^{g_i^+}$ contains a triple ramification point, then $k_i^+ - \nu_i^+ + k_i^+ - k_i^+ + 2 + k_i^+ - m_i^+ = 2k_i^+ + 2g_i^+ - 2$, and $m_i^+ + \nu_i^+ + 2g_i^+ = 4$. Thus $(m_i^+, \nu_i^+, g_i^+) =$ either $(3, 1, 0)$ or $(2, 2, 0)$ or $(1, 3, 0)$ or $(1, 1, 1)$.

(B) If $\Sigma^{g_i^+}$ does not contain any triple ramification points, then $m_i^+ + \nu_i^+ + 2g_i^+ = 2$. Thus $(m_i^+, \nu_i^+, g_i^+) = (1, 1, 0)$, since $m_i^+ \geq 1$, $\nu_i^+ \geq 1$, $g_i^+ \geq 0$.

Considering $\sum_{i=1}^{c^+} m_i^+ = m$, $\Sigma^{g^+} = \bigcup_{i=1}^{c^+} \Sigma^{g_i^+}$, (A), and (B), we have

$$\begin{cases} \nu = \displaystyle\sum_{i=1}^{c^+} \nu_i = m - 2 \text{ or } m \text{ or } m + 2 \text{ or } m, \\ c^+ = m - 2 \text{ or } m - 1 \text{ or } m \text{ or } m, \\ g^+ = \displaystyle\sum_{i=1}^{c^+} g_i^+ - c^+ + 1 = 3 - m \text{ or } 2 - m \text{ or } 1 - m \text{ or } 2 - m. \end{cases}$$

Proposition. *For $\overline{M}_+ = S^2$, (A) if $\Sigma^{g_i^+}$ contains the fixed triple branch point T, then the holomorphic map $f_i^+ : \Sigma^{g_i^+} \to \overline{M}_+$ has one of the following types:*

(1) $(\alpha_i, \alpha_j, \alpha_k; 3, 1, \dots, 1; \alpha_i + \alpha_j + \alpha_k), \quad 1 \le i < j < k \le m,$

(2) $(\alpha_i, \alpha_j; 3, 1, \dots, 1; \rho, \alpha_i + \alpha_j - \rho), \quad 1 \le \rho \le \left[\frac{\alpha_i + \alpha_j}{2}\right],$

(3) $(\alpha_i; 3, 1, \dots, 1; \rho_1, \rho_2, \alpha_i - \rho_1 - \rho_2), \quad 1 \le \rho_1 \le \rho_2 \le \left[\frac{\alpha_i + 2}{3}\right],$

(4) $(\alpha_i; 3, 1, \dots, 1; \alpha_i), \quad 1 \le i \le m,$

(B) *if $\Sigma^{g_i^+}$ contains the fixed triple branch point T, then the holomorphic map $f_i^+ : \Sigma^{g_i^+} \to \overline{M}_+$ has one of the following types:*

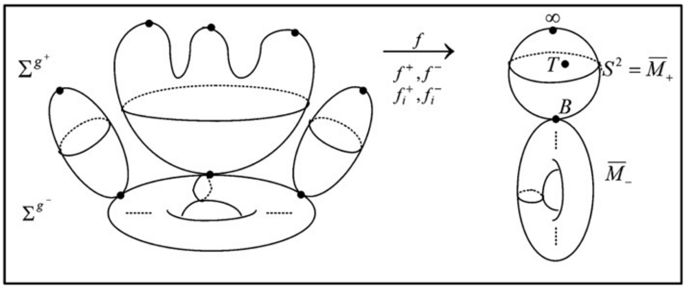
(5) $(\alpha_i; 1, 1, \dots, 1; \alpha_i), \quad 1 \le i \le m,$

at infinity, the fixed triple branch point T, and the symplectic reduction point B, respectively.

(1) If $\nu = m - 2$, then

$$\begin{cases} c^+ = m - 2, \\ g^+ = \sum_{i=1}^{c^+} g_i^+ - c^+ + 1 = 3 - m, \\ g = g^+ + g^- + \nu - 1 = g^-, \quad \text{and} \\ g^- = \sum_{i=1}^{c^-} g_i^- - c^- + 1 \le g^- - c^- + 1 = g^- - c^- + 1 \Rightarrow c^- = 1. \end{cases}$$

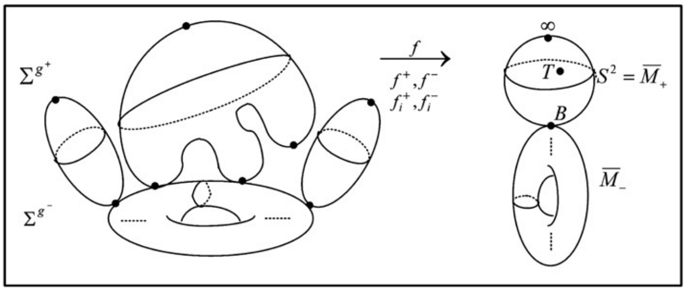
Thus, in this case f is as follows:



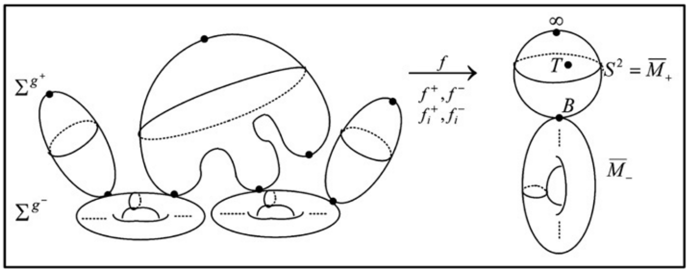
(2) If $\nu = m$, then

$$\begin{cases} c^+ = m - 1, \\ g^+ = \sum_{i=1}^{c^+} g_i^+ - c^+ + 1 = 2 - m, \\ g = g^+ + g^- + \nu - 1 = g^- + 1, \quad \text{and} \\ g^- = \sum_{i=1}^{c^-} g_i^- - c^- + 1 \le g^- - c^- + 2 \Rightarrow 1 \le c^- \le 2. \end{cases}$$

(2a) If $c^- = 1$, then, in this case f is as follows:



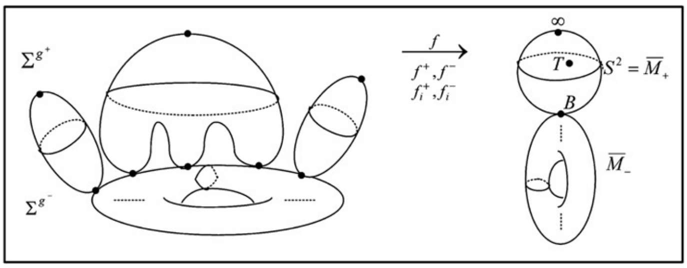
(2b) If $c^- = 2$, then, in this case f is as follows:



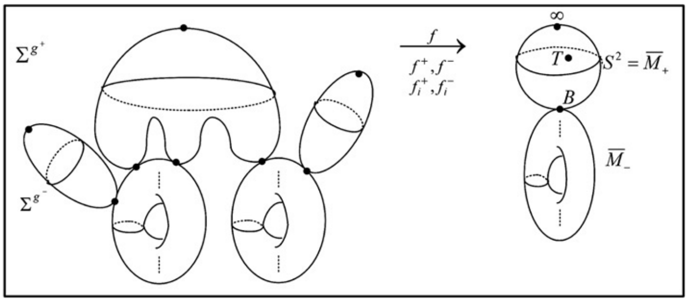
(3) If $\nu = m + 2$, then

$$\begin{cases} c^+ = m, \\ g^+ = \sum_{i=1}^{c^+} g_i^+ - c^+ + 1 = 1 - m, \\ g = g^+ + g^- + \nu - 1 = g^- + 2, \quad \text{and} \\ g^- = \sum_{i=1}^{c^-} g_i^- - c^- + 1 \leq g^- + 2 - c^- + 1 \Rightarrow 1 \leq c^- \leq 3. \end{cases}$$

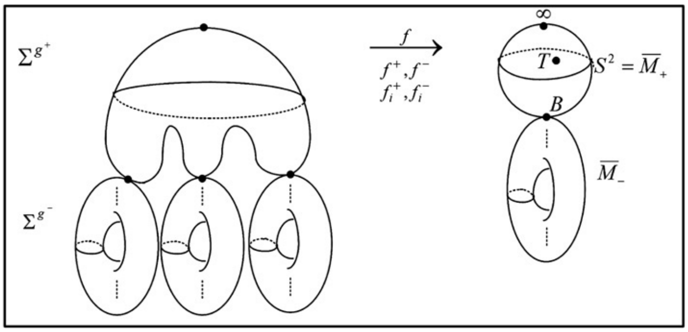
(3a) If $c^- = 1$, then, in this case f is as follows:



(3b) If $c^- = 2$, then, in this case f is as follows:



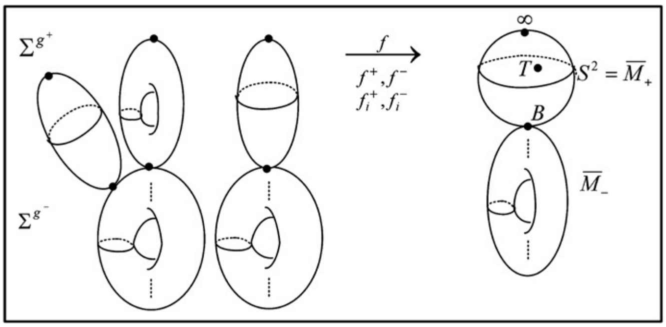
(3c) If $c^- = 3$, then, in this case f is as follows:



(4) If some $\Sigma^{g_i^+} = T^2$ is a torus, then

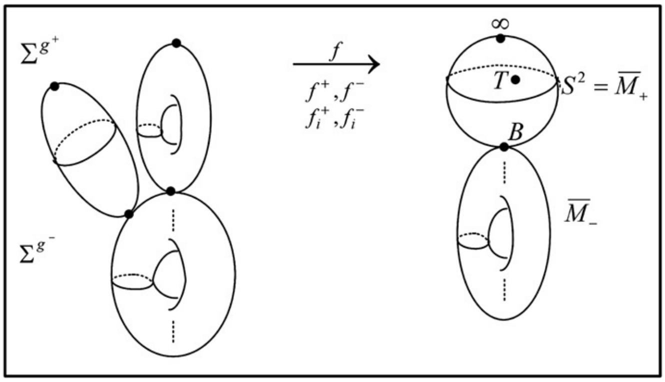
$$\begin{cases} (m_i^+, \nu_i^+, g_i^+) = (1, 1, 1), \qquad \sum_{i=1}^{c^+} m_i^+ = m = \nu = c^+, \\ g^+ = \sum_{i=1}^{c^+} g_i^+ - c^+ + 1 = 2 - m, \\ g = g^+ + g^- + \nu - 1 = g^- + 1, \quad \text{and} \\ g^- = \sum_{i=1}^{c^-} g_i^- - c^- + 1 \le g^- + 1 - c^- + 1 \Rightarrow 1 \le c^- \le 2. \end{cases}$$

(4a) If $c^- = 1$, then, in this case f is as follows:



$g = g^- + 1 = g_1^- + g_2^- - 1 + 1 = g_1^- + g_2^-$. Thus we may drop (4a) in the Hurwitz number since Σ^g is not connected.

(4b) If $c^- = 1$, then, in this case f is as follows:



Proposition. *The genus g^- and the number c^- of connected components of Σ^{g^-} are one of the following types:*

(1) $c^- = 1, \quad g^- = g, \quad \nu = m - 2$

(2) (2a) $c^- = 1, \quad g^- = g - 1, \quad \nu = m$
(2b) $c^- = 2, \quad g^- = g - 1, \quad \nu = m$

(3) (3a) $c^- = 1, \quad g^- = g - 2, \quad \nu = m + 2$
(3b) $c^- = 2, \quad g^- = g - 2, \quad \nu = m + 2$
(3c) $c^- = 3, \quad g^- = g - 2, \quad \nu = m + 2$

(4) (4b) $c^- = 1, \quad g^- = g - 1, \quad \nu = m.$

[4.3] Regarding the symplectic reduction point $B \in \overline{M}_-$ as ∞, we get many new 3-simple ramified covering maps

$$f_i^- : \Sigma^{g_i^-} \longrightarrow \overline{M}_- = \Sigma^h.$$

However, in the above cases, the holomorphic map f_i^- has either a strictly smaller number of ramification points at infinity (the case (1)) or a strictly smaller degree (2b, 3b, 3c) or a strictly smaller genus (the cases (2)–(4)) than the holomorphic map $f : \Sigma^g \to \Sigma^h$.

Thus, if we know the relative GW-invariant in $\overline{M}_+$, then as [16], we can get a recursive formula for $\mu_{h,m}^{g,k,3}(\alpha)$.

[4.4] Let us consider the following types:

$$\theta_1 = (\alpha_1, \ldots, \hat{\alpha_i}, \ldots, \hat{\alpha_j}, \ldots \hat{\alpha_k}, \ldots, \alpha_m, \alpha_i + \alpha_j, \alpha_k),$$
$$\theta_2 = (\alpha_1, \ldots, \hat{\alpha_i}, \ldots, \hat{\alpha_j}, \ldots, \alpha_m, \rho, \alpha_i + \alpha_j - \rho),$$
$$\theta_3 = (\alpha_1, \ldots, \hat{\alpha_i}, \ldots, \rho_1, \rho_2, \alpha_i - \rho_1 - \rho_2).$$

Let $J_1(\alpha)$ be the set of ordered $(m-2)$-tuples $\theta_1 = (\alpha_1, \ldots, \hat{\alpha_i}, \ldots, \hat{\alpha_j}, \ldots, \hat{\alpha_k}, \ldots, \alpha_m, \alpha_i + \alpha_j, \alpha_k)$, where we have the identity $\theta_1 \sim \theta_1'$ iff $\{\alpha_i, \alpha_j, \alpha_k\} = \{\alpha_i', \alpha_j', \alpha_k'\}$.

For any $\theta_1 = (\alpha_1, \ldots, \hat{\alpha_i}, \ldots, \hat{\alpha_j}, \ldots \hat{\alpha_k}, \ldots, \alpha_m, \alpha_i + \alpha_j + \alpha_k) \in J_1(\alpha)$, we define an integer:

$$I_1(\theta_1) = \begin{cases} \frac{1}{6}(\alpha_i + \alpha_j + \alpha_k) \cdot \sharp\{\lambda \in \theta_1 | \lambda = \alpha_i + \alpha_j + \alpha_k\} & \text{if } \alpha_i = \alpha_j = \alpha_k, \\ \frac{1}{2}(\alpha_i + \alpha_j + \alpha_k) \cdot \sharp\{\lambda \in \theta_1 | \lambda = \alpha_i + \alpha_j + \alpha_k\} & \text{if either } \alpha_i = \alpha_j \text{ or } \alpha_j = \alpha_k, \\ (\alpha_i + \alpha_j + \alpha_k) \cdot \sharp\{\lambda \in \theta_1 | \lambda = \alpha_i + \alpha_j + \alpha_k\} & \text{if } \alpha_i, \alpha_j, \alpha_k \text{ are distinct.} \end{cases}$$

For any pair (α_i, α_j) in $\alpha = (\alpha_1, \ldots, \alpha_m)$, we set $C_{ij}(\alpha) = \Big\{\theta_2 = (\alpha_1, \ldots, \hat{\alpha_i}, \ldots, \hat{\alpha_j}, \ldots, \alpha_m, \rho + \alpha_i + \alpha_j - \rho)$ $|1 \le \rho \le \left[\frac{\alpha_i + \alpha_j}{2}\right]\Big\}$ of ordered m-tuples. Let $A = \{(\alpha_i, \alpha_j)\}$ be the distinct pairs of elements in α.

Let $C(\alpha) := \bigcup_{(\alpha_i,\alpha_j)\in A} C_{ij}(\alpha)$. For every $\theta_2 \in C(\alpha)$, we associate with it an integer:

$$I_2(\theta_2) = \begin{cases} \frac{1}{2}\rho(\alpha_i+\alpha_j-\rho)\cdot \sharp\{\lambda\in\theta_2|\lambda=\rho\}\cdot (\sharp\{\lambda\in\theta_2|\lambda=\alpha_i-\rho\}-1) & \text{if } 2\rho=\alpha_i,\\ \rho(\alpha_i+\alpha_j-\rho)\cdot \sharp\{\lambda\in\theta_2|\lambda=\rho\}\cdot \sharp\{\lambda\in\theta_2|\lambda=\alpha_i-\rho\} & \text{if } 2\rho\neq\alpha_i. \end{cases}$$

Dividing $\{1,2,\ldots,\hat{i},\ldots,\hat{j},\ldots,m\}$ into two parts π_1,π_2, correspondingly, we divide $\theta_2=(\alpha_1,\ldots,\hat{\alpha_i},\ldots,\hat{\alpha_j},\ldots,\alpha_m,\rho,\alpha_i+\alpha_j-\rho)$ into two parts in forms: $\theta_{2,\pi_1}=(\alpha_{\pi_1},\rho)$, $\theta_{2,\pi_2}=(\alpha_{\pi_2},\alpha_i+\alpha_j-\rho)$.

Denote $P_{\theta_2}=\{\pi=(\theta_{2,\pi_1},\theta_{2,\pi_2})\}/\sim$, where $\pi=(\theta_{2,\pi_1},\theta_{2,\pi_2})=(\ldots,\rho,\ldots,\alpha_i+\alpha_j-\rho)\sim\pi'=(\theta_{2,\pi_{1'}},\theta_{2,\pi_{2'}})=(\ldots,\rho',\ldots,\alpha_i+\alpha_j-\rho')$ if $\rho=\rho'$ and θ_{2,π_1} and $\theta_{2,\pi_{1'}}$ are the same through a permutation.

For any $\pi=(\theta_{2,\pi_1},\theta_{2,\pi_2})\in P_{\theta_2}$, we associate with it an integer:

$$I_3(\pi) = \begin{cases} \frac{1}{2}\rho(\alpha_i+\alpha_j-\rho)\cdot \sharp\{\lambda\in\theta_{2,\pi_1}|\lambda=\rho\}\cdot \sharp\{\lambda\in\theta_{2,\pi_2}|\lambda=\alpha_i+\alpha_j-\rho\} & \text{if } 2\rho=\alpha_i+\alpha_j,\\ \rho(\alpha_i+\alpha_j-\rho)\cdot \sharp\{\lambda\in\theta_{2,\pi_1}|\lambda=\rho\}\cdot \sharp\{\lambda\in\theta_{2,\pi_2}|\lambda=\alpha_i+\alpha_j-\rho\} & \text{if } 2\rho\neq\alpha_i+\alpha_j. \end{cases}$$

For every α_i in α, we construct a set $D_{\alpha_i}(\alpha)=\left\{\theta_3=(\alpha_1,\ldots,\hat{\alpha}_i,\ldots,\alpha_m,\rho_1,\rho_2,\alpha_i-\rho_1-\rho_2)|1\le\rho_1\le\rho_2\le\left[\frac{\alpha_i+2}{3}\right]\right\}$ of ordered $(m+2)$-tuples. Let $\alpha_{1'},\ldots,\alpha_{l'}$ be the distinct elements in α. Let $D(\alpha)=D_{\alpha_{1'}}(\alpha)\cup\cdots\cup D_{\alpha_{l'}}(\alpha)$.

For any $\theta_3\in D(\alpha)$, we associate with it an integer:

$$I_4(\theta_3) = \begin{cases} \frac{1}{6}\rho_1\cdot\rho_2\cdot(\alpha_i-\rho_1-\rho_2)\cdot\sharp\{\lambda\in\theta_3|\lambda=\rho_1\}\\ \quad\cdot(\sharp\{\lambda\in\theta_3|\lambda=\rho_2\}-2)\cdot(\sharp\{\lambda\in\theta_3|\lambda=\alpha_i-\rho_1-\rho_2\}-2) & \text{if } \rho_1=\rho_2=\alpha_i-\rho_1-\rho_2,\\ \frac{1}{2}\rho_1\cdot\rho_2\cdot(\alpha_i-\rho_1-\rho_2)\cdot(\sharp\{\lambda\in\theta_3|\lambda=\rho_1\}-1)\\ \quad\cdot\sharp\{\lambda\in\theta_3|\lambda=\rho_2\}\cdot\sharp\{\lambda\in\theta_3|\lambda=\alpha_i-\rho_1-\rho_2\} & \text{if } \rho_1=\rho_2\neq\alpha_i-\rho_1-\rho_2,\\ \frac{1}{2}\rho_1\cdot\rho_2\cdot(\alpha_i-\rho_1-\rho_2)\cdot\sharp\{\lambda\in\theta_3|\lambda=\rho_1\}\\ \quad\cdot(\sharp\{\lambda\in\theta_3|\lambda=\rho_2\}-1)\cdot\sharp\{\lambda\in\theta_3|\lambda=\alpha_i-\rho_1-\rho_2\} & \text{if } \rho_1\neq\rho_2=\alpha_i-\rho_1-\rho_2,\\ \frac{1}{2}\rho_1\cdot\rho_2\cdot(\alpha_i-\rho_1-\rho_2)\cdot\sharp\{\lambda\in\theta_3|\lambda=\rho_1\}\\ \quad\cdot\sharp\{\lambda\in\theta_3|\lambda=\rho_2\}\cdot(\sharp\{\lambda\in\theta_3|\lambda=\alpha_i-\rho_1-\rho_2\}-1) & \text{if } \rho_2\neq\rho_1=\alpha_i-\rho_1-\rho_2,\\ \rho_1\cdot\rho_2\cdot(\alpha_i-\rho_1-\rho_2)\cdot\sharp\{\lambda\in\theta_3|\lambda=\rho_1\}\\ \quad\cdot\sharp\{\lambda\in\theta_3|\lambda=\rho_2\}\cdot\sharp\{\lambda\in\theta_3|\lambda=\alpha_i-\rho_1-\rho_2\} & \text{if } \rho_1,\rho_2,\alpha_i-\rho_1-\rho_2 \text{ are distinct.} \end{cases}$$

Dividing $\{1,2,\ldots,\hat{i},\ldots,m\}$ into two parts π_1,π_2, correspondingly, we divide $\theta_3=(\alpha_1,\ldots,\hat{\alpha}_i,\ldots,\alpha_m,\rho_1,\rho_2,\alpha_i-\rho_1-\rho_2)$ into two parts in three forms:

$$\begin{aligned} \pi &= (\theta_{3,\pi_1};\theta_{3,\pi_2}) = (\alpha_{\pi_1},\rho_1,\rho_2;\alpha_{\pi_2},\alpha_i-\rho_1-\rho_2),\\ \pi' &= (\theta'_{3,\pi_1};\theta'_{3,\pi_2}) = (\alpha_{\pi_1},\rho_1,\alpha_i-\rho_1-\rho_2;\alpha_{\pi_2},\rho_2),\\ \pi'' &= (\theta''_{3,\pi_1};\theta''_{3,\pi_2}) = (\alpha_{\pi_1},\rho_2,\alpha_i-\rho_1-\rho_2;\alpha_{\pi_2},\rho_1). \end{aligned}$$

Let $P_{\theta_3}=\{\pi=(\theta_{3,\pi_1};\theta_{3,\pi_2}),\pi',\pi''\}/\sim$, where $\pi\sim\tilde{\pi}$ iff $\{\rho_1,\rho_2\}=\{\tilde{\rho}_1,\tilde{\rho}_2\}$, and θ_{3,π_1} and $\theta_{3,\tilde{\pi}_1}$ are the same through a permutation. For any $\pi\in P_{\theta_3}$, we associate with it an integer:

$$I_5(\theta_3) = \begin{cases} \frac{1}{4}\rho_1\cdot\rho_2\cdot(\alpha_i-\rho_1-\rho_2)\cdot\sharp\{\lambda\in\theta_{3,\pi_1}|\lambda=\rho_1\}\\ \quad\cdot(\sharp\{\lambda\in\theta_{3,\pi_1}|\lambda=\rho_2\}-1)\cdot\sharp\{\lambda\in\theta_{3,\pi_2}|\lambda=\alpha_i-\rho_1-\rho_2\} & \text{if } \rho_1=\rho_2=\alpha_i-\rho_1-\rho_2,\\ \frac{1}{2}\rho_1\cdot\rho_2\cdot(\alpha_i-\rho_1-\rho_2)\cdot\sharp\{\lambda\in\theta_{3,\pi_1}|\lambda=\rho_1\}\\ \quad\cdot(\sharp\{\lambda\in\theta_{3,\pi_1}|\lambda=\rho_2\}-1)\cdot\sharp\{\lambda\in\theta_{3,\pi_2}|\lambda=\alpha_i-\rho_1-\rho_2\} & \text{if } \rho_1=\rho_2\neq\alpha_i-\rho_1-\rho_2,\\ \frac{1}{2}\rho_1\cdot\rho_2\cdot(\alpha_i-\rho_1-\rho_2)\cdot\sharp\{\lambda\in\theta_{3,\pi_1}|\lambda=\rho_1\}\\ \quad\cdot\sharp\{\lambda\in\theta_{3,\pi_1}|\lambda=\rho_2\}\cdot\sharp\{\lambda\in\theta_{3,\pi_2}|\lambda=\alpha_i-\rho_1-\rho_2\} & \text{if } \rho_1\neq\rho_2,\ \rho_1 \text{ or } \rho_2=\alpha_i-\rho_1-\rho_2,\\ \rho_1\cdot\rho_2\cdot(\alpha_i-\rho_1-\rho_2)\cdot\sharp\{\lambda\in\theta_{3,\pi_1}|\lambda=\rho_1\}\\ \quad\cdot\sharp\{\lambda\in\theta_{3,\pi_1}|\lambda=\rho_2\}\cdot\sharp\{\lambda\in\theta_{3,\pi_2}|\lambda=\alpha_i-\rho_1-\rho_2\} & \text{if } \rho_1,\rho_2,\alpha_i-\rho_1-\rho_2 \text{ are distinct.} \end{cases}$$

Dividing $\{1, \ldots, \hat{i}, \ldots, m\}$ into 3 parts π_1, π_2 and π_3, correspondingly, we divide $\theta_3 = (\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_m, \rho_1, \rho_2, \alpha_i - \rho_1, \rho_2)$ into 3 parts in forms: $\theta_3 = (\theta_{3,\pi_1}; \theta_{3,\pi_2}; \theta_{3,\pi_3}) = (\alpha_{\pi_1}, \rho_1; \alpha_{\pi_2}, \rho_2; \alpha_{\pi_3}, \alpha_i - \rho_1 - \rho_2)$.

Let $\bar{P}_{\theta_3} = \{\pi = (\theta_{3,\pi_1}; \theta_{3,\pi_2}; \theta_{3,\pi_3})\}/\sim$, where $\pi \sim \tilde{\pi}$ iff $\{\rho_1, \rho_2\} = \{\tilde{\rho}_1, \tilde{\rho}_2\}$ and $\{\theta_{3,\pi_1}; \theta_{3,\pi_2}\}$ and $\{\theta_{3,\tilde{\pi}_1}; \theta_{3,\tilde{\pi}_2}\}$ are the same through permutations.

For any $\pi \in \bar{P}_{\theta_3}$, we associate with it an integer:

$$I_6(\pi) = \begin{cases} \frac{1}{6}\rho_1 \cdot \rho_2 \cdot (\alpha_i - \rho_1 - \rho_2) \cdot \sharp\{\lambda \in \theta_{3,\pi_1} | \lambda = \rho_1\} \\ \quad \cdot \sharp\{\lambda \in \theta_{3,\pi_2} | \lambda = \rho_2\} \cdot \sharp\{\lambda \in \theta_{3,\pi_3} | \lambda = \alpha_i - \rho_1 - \rho_2\} \quad \text{if } \rho_1 = \rho_2 = \alpha_i - \rho_1 - \rho_2, \\ \frac{1}{2}\rho_1 \cdot \rho_2 \cdot (\alpha_i - \rho_1 - \rho_2) \cdot \sharp\{\lambda \in \theta_{3,\pi_1} | \lambda = \rho_1\} \cdot \sharp\{\lambda \in \theta_{3,\pi_2} | \lambda = \rho_2\} \\ \quad \cdot \sharp\{\lambda \in \theta_{3,\pi_3} | \lambda = \alpha_i - \rho_1 - \rho_2\} \quad \text{if only two in } \{\rho_1, \rho_2, \alpha_i - \rho_1 - \rho_2\} \text{ are equal}, \\ \rho_1 \cdot \rho_2 \cdot (\alpha_i - \rho_1 - \rho_2) \cdot \sharp\{\lambda \in \theta_{3,\pi_1} | \lambda = \rho_1\} \\ \quad \cdot \sharp\{\lambda \in \theta_{3,\pi_2} | \lambda = \rho_2\} \cdot \sharp\{\lambda \in \theta_{3,\pi_3} | \lambda = \alpha_i - \rho_1 - \rho_2\} \quad \text{if } \rho_1, \rho_2, \alpha_i - \rho_1 - \rho_2 \text{ are distinct.} \end{cases}$$

[4.5] Let us compute several distinct GW-invariants

Proposition. (1) *The product of the relative GW-invariants of* $(m-3)$ *components in* θ_1 *is*

$$\psi(\alpha, \theta_1) = \begin{cases} \frac{1}{\alpha_1} \cdots \frac{\hat{1}}{\alpha_i} \cdots \frac{\hat{1}}{\alpha_j} \cdots \frac{\hat{1}}{\alpha_k} \cdots \frac{1}{\alpha_m} & \text{if } \alpha_i, \alpha_j, \alpha_k \text{ are distinct}, \\ \frac{1}{2}\frac{1}{\alpha_1} \cdots \frac{\hat{1}}{\alpha_i} \cdots \frac{\hat{1}}{\alpha_j} \cdots \frac{\hat{1}}{\alpha_k} \cdots \frac{1}{\alpha_m} & \text{if either } \alpha_i = \alpha_j \text{ or } \alpha_j = \alpha_k, \\ \frac{1}{6}\frac{1}{\alpha_1} \cdots \frac{\hat{1}}{\alpha_i} \cdots \frac{\hat{1}}{\alpha_j} \cdots \frac{\hat{1}}{\alpha_k} \cdots \frac{1}{\alpha_m} & \text{if } \alpha_i = \alpha_j = \alpha_k. \end{cases}$$

(2) *The product of the relative GW-invariants of* $(m-2)$ *components in* θ_2 *is*

$$\psi(\alpha, \theta_2) = \begin{cases} \frac{1}{\alpha_1} \cdots \frac{\hat{1}}{\alpha_i} \cdots \frac{1}{\alpha_m} & \text{if } \alpha_i + \alpha_j = 2\rho, \\ \frac{1}{2}\frac{1}{\alpha_1} \cdots \frac{\hat{1}}{\alpha_i} \cdots \frac{1}{\alpha_m} & \text{if } \alpha_i + \alpha_j \neq 2\rho. \end{cases}$$

(3) *The product of the relative GW-invariants of* $(m-1)$ *components in* θ_3 *is*

$$\psi(\alpha, \theta_3) = \begin{cases} \frac{1}{\alpha_1} \cdots \frac{\hat{1}}{\alpha_i} \cdots \frac{1}{\alpha_m} & \text{if } \rho_1, \rho_2, \alpha_i - \rho_1 - \rho_2 \text{ are distinct}, \\ \frac{1}{2}\frac{1}{\alpha_1} \cdots \frac{\hat{1}}{\alpha_i} \cdots \frac{1}{\alpha_m} & \text{if either } \rho_1 = \rho_2 \text{ or } \rho_2 = \alpha_i - \rho_1 - \rho_2, \\ \frac{1}{6}\frac{1}{\alpha_1} \cdots \frac{\hat{1}}{\alpha_i} \cdots \frac{1}{\alpha_m} & \text{if } \rho_1 = \rho_2 = \alpha_i - \rho_1 - \rho_2. \end{cases}$$

Proof. Compute the relative GW-invariants in connected components when $\overline{M}_+ = S^2$:

$$\psi^{S^2,\mathrm{pt,pt,pt}}_{kH,0,0}\left(|_{\mathrm{pt,pt,pt}}; k; 1, \ldots, 1;\ k\right) = \frac{1}{k},$$

$$\psi^{S^2,\mathrm{pt,pt,pt}}_{kH,0,0}\left(|_{\mathrm{pt,pt,pt}}; k; 3, 1, \ldots, 1; \rho_1, \rho_2, k - \rho_1 - \rho_2\right) = \begin{cases} \frac{1}{6} & \text{if } \rho_1 = \rho_2 = k - \rho_1 - \rho_2 \\ \frac{1}{2} & \text{if either } \rho_1 = \rho_2 \text{ or } \rho_2 = k - \rho_1 - \rho_2 \\ 1 & \text{if } \rho_1, \rho_2, k - \rho_1 - \rho_2 \text{ are distinct,} \end{cases}$$

$$\psi^{S^2,\mathrm{pt,pt,pt}}_{kH,0,0}(|_{\mathrm{pt,pt,pt}}; k_1, k_2; 3, 1, \ldots, 1; k_3, k_4) = \begin{cases} \frac{1}{4} & \text{if } k_1 = k_2 \text{ and } k_3 = k_4 \\ \frac{1}{2} & \text{if either } k_1 = k_2 \text{ or } k_3 = k_4 \\ 1, & \text{if } k_1 \neq k_2 \text{ and } k_3 \neq k_4 \end{cases} \tag{1}$$

where $k_1 + k_2 = k_3 + k_4 = k$,

$$\psi^{T^2,\mathrm{pt},\mathrm{pt},\mathrm{pt}}_{kH,0,0}\left(|_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}; k; 3, 1, \ldots, 1; k\right) = \mu^{1,k,3}_{0,3}(k).$$

Considering the following maps $F_1, F_2, F_3 : S^2 \to S^2$ given by

$$F_1(z) = z^k,$$
$$F_2(z) = \frac{(z - y_1)^{\rho_1}(z - y_2)^{\rho_2}(z - y_3)^{k-\rho_1-\rho_2}}{z^k},$$
$$F_3(z) = \frac{(z - y_4)^{\rho_1}(z - y_5)^{k-\rho_1}}{(z - y_6)^{\rho_2}(z - y_7)^{k-\rho_2}},$$

we can get the above.

Let $F_4 : T^2 = S^1 \times S^1 \to S^2$ be a 3-simple ramified covering map of type $(k; 3, 1, \ldots, 1; k)$.

The dimension of the moduli space $\overline{\mathcal{M}}^{S^2,Z}_{kH,0}(1, k)$ is $2c_1(S^2)kH + (4g - 4) + 3\{1 + (k - 2) + 1\} - 2 \cdot 3k = 4k + 2k - 6k = 0$.

Thus we have

$$\psi^{T^2,\mathrm{pt},\mathrm{pt},\mathrm{pt}}_{kH,0,0}\left(|_{\mathrm{pt},\,\mathrm{pt},\mathrm{pt}}; k; 3, 1, \ldots, 1; k\right) = \mu^{1,k,3}_{0,3}(k) = \sharp\overline{\mathcal{M}}^{S^2,Z}_{kH,0}(1, k). \quad \blacksquare$$

Summing up all of the above, we have our main theorem:

Theorem. *All Hurwitz numbers $\mu^{g,k,3}_{h,m}(\alpha)$ of 3-simple ramified coverings can be determined by the following recursive formula:*

$$\begin{aligned}
\mu^{g,k,3}_{h,m}(\alpha) &= \sum_{\theta_1\in J_1(\alpha)} \mu^{g,k,3}_{h,m-2}(\theta_1)\cdot I_1(\theta_1) + \sum_{\theta_2\in C(\alpha)} \mu^{g-1,k,3}_{h,m}(\theta_2)\cdot I_2(\theta_2)\\
&\quad + \sum_{\theta_2\in C(\alpha)} \sum_{\substack{g_1+g_2=g\\ g_1,g_2\ge 0}} \sum_{\substack{k_1+k_2=k\\ k_1,k_2\ge 1}} \sum_{\substack{m_1+m_2=m\\ m_1,m_2\ge 1}} \sum_{\pi\in P_{\theta_2}} \begin{pmatrix} \frac{1}{2}(k+m) - kh + g - 2\\ \frac{1}{2}(k_1+m_1) - k_1h + g_1 - 1\end{pmatrix}\\
&\quad \times \mu^{g_1,k_1,3}_{h,m_1}(\theta_{2,\pi_1})\cdot \mu^{g_2,k_2,3}_{h,m_2}(\theta_{2,\pi_2})\cdot I_3(\pi) + \sum_{\theta_3\in D(\alpha)} \mu^{g-2,k,3}_{h,m+2}(\theta_3)\cdot I_4(\theta_3)\\
&\quad + \sum_{\theta_3\in D(\alpha)} \sum_{\substack{g_1+g_2=g-1\\ g_1,g_2\ge 0}} \sum_{\substack{k_1+k_2=k\\ k_1,k_2\ge 1}} \sum_{\substack{m_1+m_2=m+2\\ m_1,m_2\ge 1}} \sum_{\pi\in P_{\theta_3}} \begin{pmatrix} \frac{1}{2}(k+m) - kh + g - 2\\ \frac{1}{2}(k_1+m_1) - k_1h + g_1 - 1\end{pmatrix}\\
&\quad \times \mu^{g_1,k_1,3}_{h,m_1}(\theta_{3,\pi_1})\cdot \mu^{g_2,k_2,3}_{h,m_2}(\theta_{3,\pi_2})\cdot I_5(\pi)\\
&\quad + \sum_{\theta_3\in D(\alpha)} \sum_{\substack{g_1+g_2+g_3=g\\ g_1,g_2,g_3\ge 0}} \sum_{\substack{k_1+k_2+k_3=k\\ k_1,k_2,k_3\ge 1}} \sum_{\substack{m_1+m_2+m_3=m+2\\ m_1,m_2,m_3\ge 1}} \sum_{\pi\in \bar{P}_{\theta_3}} \begin{pmatrix} \frac{1}{2}(k+m) - kh + g - 2\\ \frac{1}{2}(k_1+m_1) - k_1h + g_1 - 1\end{pmatrix}\\
&\quad \times \begin{pmatrix} N\\ \frac{1}{2}(k_2+m_2) - k_2h + g_2 - 1\end{pmatrix} \mu^{g_1,k_1,3}_{h,m_1}(\theta_{3,\pi_1})\cdot \mu^{g_2,k_2,3}_{h,m_2}(\theta_{3,\pi_2})\cdot \mu^{g_3,k_3,3}_{h,m_3}(\theta_{3,\pi_3})\cdot I_6(\pi)\\
&\quad + \mu^{1,k,3}_{0,3}(k; 3, 1, \ldots, 1; k)\cdot \mu^{g_1,k,3}_{h,m}(\alpha),
\end{aligned}$$

where $N = [\frac{1}{2}(k + m) - kh + g - 2] - [\frac{1}{2}(k_1 + m_1) - k_1h + g_1 - 1]$.

Remark. The initial values of our recursive formula are

$$\mu^{g,1}_{0,1}(1) = \begin{cases} 1 & \text{if } g = 0\\ 0 & \text{if } g \ge 1,\end{cases}$$

$$\mu_{h,1}^{g,1}(1) = \begin{cases} 1 & \text{if } g = h > 0 \\ 0 & \text{if } g \geq h+1, \end{cases}$$

and some initial values when $k + m - 2kh + 2g - 2 = 0$.

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